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LONGITUDINAL DIFFUSION OF SOLID PARTICLE ADMIXTURE IN A FLOW THROUGH A TUBE

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Abstract—The propagation of solid particle admixture in a flow through a flat channel is studied. The processes of diffusion and convective transfer as well as solid particle deposition due to gravity result in varying admixture concentration both in depth and longitudinally.

The study of admixture longitudinal distribution is of great interest in a lot of applications, therefore this paper gives the derivation of longitudinal diffusion equation for a mean cross-section admixture concentration.

The equation contains three effective parameters; i.e. convective transfer velocity, longitudinal diffusion coefficient and particle deposition time. These parameters integrally reflect local processes of matter transfer as well as momentum.

The proposed model is specific and differs from Taylor equation for longitudinal diffusion, since the fact of particle deposition and adhesion is taken into account. As a result of particle deposition a sediment layer is formed on the channel bottom which increases in thickness with time. To describe this process balance conditions for the whole flow mass and admixture mass on sediment surface are formulated and a condition for matter movement towards the channel bottom is derived that is different from zero due to particle adhesion.

1. INTRODUCTION

In the 1950s Taylor has formulated a problem, i.e. for three-dimensional equation of matter transfer in a tube or a channel to derive one-dimensional equation for longitudinal diffusion the solution of which properly describes a distribution of average particle concentration.

The description of the transfer process based on a longitudinal diffusion equation significantly simplifies calculations for a research and allows to get sufficient information on admixture distribution.

The field of application of the longitudinal diffusion equation is very wide, therefore the solution of Taylor problem is of great scientific and practical significance.

Taylor (1953, 1954) has given a classical example of solving the transfer problem of "passive" admixture in a circular tube at $t \to \infty$ and initial concentration distribution—$C_o(x)$.

Taylor's works served as methodical base and a starting point for multiple research in this direction.

The proposed one-dimensional diffusion model refers to flows in which liquid and solid particles have different velocities. Besides, the solid particles which have reached the lower surface of a tube, are assumed to adhere to it and not to participate in a movement any more.

The longitudinal diffusion equation derived here, significantly differs from Taylor equation as well as Sayre (1969) and Sümer (1971) equations.

More recently an original contribution on the subject has been made by Smith (1983).

2. HARD PARTICLE ADMIXTURE TRANSFER EQUATION

Assume that liquid and admixture particles fill the flow region completely and represent two continuums with densities ρ_i $(i = 1, 2)$.

Consider a case when a flow contains a large quantity of particles but their volume concentration is small $s \ll 1$. Assume that sizes of solid particles are small compared with the characteristic linear scales of turbulence.

Consider a suspension flow as two-phase system. The continuity equations for each phase may be written as:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 v_{1\alpha}}{\partial x_\alpha} = 0,$$
$$\frac{\partial \rho_2}{\partial t} + \frac{\partial \rho_2 v_{2\alpha}}{\partial x_\alpha} = 0, \quad \alpha = 1, 2, 3 \qquad [1]$$

where ρ_1 and ρ_2 = continuum densities of liquid and solid particles; $v_{1\alpha}$ and $v_{2\alpha}$ = components of continuum velocities for liquid and solid particles.

Separate phase densities may be expressed in terms of volume concentration of solid particles s and true densities of liquid and solid particles, d_1 and d_2, respectively.

We have:

$$\rho_1 = (1 - s)d_1, \quad \rho_2 = sd_2. \qquad [2]$$

Then the second equation of the system [1] is following;

$$\frac{\partial s}{\partial t} + \frac{\partial s v_{2\alpha}}{\partial x_\alpha} = 0. \qquad [3]$$

According to the method of the turbulent flow description we represent flow parameters as a sum of their averaged values and pulsation components:

$$s = S + s', \; v_{i\alpha} = V_{i\alpha} + v'_{i\alpha}, \; i = 1, 2 \qquad [4]$$

where S and $V_{i\alpha}$ = averaged values of volume concentration and velocity components; s' and $v'_{i\alpha}$ = pulsation components.

Equation [3] may be rewritten as

$$\frac{\partial S}{\partial t} + \frac{\partial S(V_{2\alpha} - V_{1\alpha})}{\partial x_\alpha} + \frac{\partial S V_{1\alpha}}{\partial x_\alpha} = -\frac{\partial \overline{s' v'_{2\alpha}}}{\partial x_\alpha} \qquad [5]$$

where the difference $V_{2\alpha} - V_{1\alpha}$ is a relative averaged velocity of admixture in a flow.

Assume that the averaged horizontal continuum velocities of solid and liquid particles coincide and the vertical ones differ by a. We have:

$$V_{2\alpha} = V_{1\alpha} - a\delta_{\alpha 3}, \; (\delta_{\alpha\beta} = 0, \; \alpha \neq \beta, \; \delta_{\alpha\alpha} = 1). \qquad [6]$$

The a-value denotes particle deposition velocity in a gravity field and is called a hydraulic grain size. If solid particles of admixture have more or less equal sizes the a-value may be considered constant.

Taking into consideration [6] we get

$$\frac{\partial S}{\partial t} - a\frac{\partial S}{\partial x_3} + \frac{\partial S V_{1\alpha}}{\partial x_\alpha} = -\frac{\partial \overline{s' v'_{2\alpha}}}{\partial x_\alpha}. \qquad [7]$$

Let s'- and $v'_{2\alpha}$-parameters correlation be proportional to the S-function gradient.

$$\overline{s' v'_{2\alpha}} = \epsilon_{\alpha\beta}\frac{\partial S}{\partial x_\beta}. \qquad [8]$$

In this equality $\epsilon_{\alpha\beta}$ is a tensor of diffusion coefficients which is

$$\epsilon_{\alpha\beta} = -\epsilon\delta_{\alpha\beta} \qquad [9]$$

in the case of isotropy diffusion.

Equation [7] using [8] may be written as

$$\frac{\partial S}{\partial t} - a\frac{\partial S}{\partial x_3} + \frac{\partial SV_{1\alpha}}{\partial x_\alpha} = -\frac{\partial}{\partial x_\alpha}\left(\epsilon_{\alpha\beta}\frac{\partial S}{\partial x_\beta}\right). \qquad [10]$$

3. DIFFUSION EQUATION IN A PLANE PARALLEL FLOW

Consider transfer of admixture of solid particles in a plane parallel flow of H depth. Let Z-axis be directed vertically upwards and X-axis in the flow direction. According to the case studied the liquid particle velocity vector will have only one, different from zero, component-$V_x(z)$.

Taking this fact into consideration, [10] may be written as:

$$\frac{\partial S}{\partial t} - a\frac{\partial S}{\partial z} + V_x(z)\frac{\partial S}{\partial x} = \frac{\partial}{\partial x}\left(\epsilon\frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial z}\left(\epsilon\frac{\partial S}{\partial z}\right). \qquad [11]$$

Assume that in [11] the longitudinal flow velocity and transfer coefficient do not differ from corresponding parameters in a homogeneous flow which does not contain admixture particles. This assumption is justified to some extent by a previously assumed condition that particle volume concentration is small. In fact, the presence of solid particles changes the turbulent flow pattern and affects the mass transfer intensity. The transfer equation subject to this effect and more complex effects are derived in Barenblatt (1953) and Nikolaevskii (1963). The difficulties to find solutions of this equation are discussed in works of Buyevich (1965).

Suppose that ϵ coefficient is a constant value equal to molecular diffusion coefficient. Equation [11] in this case describes solid particle transfer process in a laminar flow.

4. BOUNDARY CONDITIONS

Assume that solid particles settle down on the bottom of the channel and form a layer of $h(t, x)$ thickness. In this layer the particles do not move and their volume concentration is equal to s_*.

We shall consider deposit layer boundary as discontinuity surface on which a number of flow parameters change abruptly:

$$\begin{gathered}[s] \neq 0,\ [\rho] \neq 0,\ [v_n] \neq 0 \\ v_n = \frac{1}{\rho}(\rho_1 v_{1n} + \rho_2 v_{2n}),\ \rho = \rho_1 + \rho_2\end{gathered} \qquad [12]$$

where ρ = mixture density; $\mathbf{n}$ = exterior normal vector to deposit layer surface; v_n-mixture velocity component normal to layer surface.

Solid particle phase flow is equal to $s^+ \cdot v_{2n}^+$, where s^+ and v_{2n}^+ are concentration value and normal component of phase velocity on the upper boundary of the layer. The corresponding values for deposit layer are $s^- = s_*$ and $v_{2n}^- = 0$.

We assume that the angle between Z-axis direction and normal vector-to-layer surface is small. Then an equation for the same flow for the accepted model[11] may be written as:

$$\left(\epsilon\frac{\partial S}{\partial z}+aS\right)_{z=h}=(v_{2z}S)_{z=h}. \qquad [13]$$

Suppose the matter layer of the second phase at the upper flow boundary not to form, then we accept the following condition

$$\left(\epsilon\frac{\partial S}{\partial z}+aS\right)_{z=H}=0. \qquad [14]$$

Let us find an equation for the thickness of a deposit layer on the channel bottom. From the considerations of the matter balance of the second phase at the interface we have

$$(s_{*}-S/_{z=h})D_n=(v_{2z}S)_{z=h},\ D_n\approx\frac{\partial h}{\partial t} \qquad [15]$$

where D_n = normal velocity of layer boundary motion. Equations [13]–[15] include two parameters, i.e. s_{*} and $v_{2n}/_{z=h}$, which can not be found within the model studied; to determine these parameters it is necessary to make use of experimental data and more detailed theory. A review of the works in this field is presented in Mednikov (1980).

5. DERIVATION OF ONE-DIMENSIONAL MATTER TRANSFER MODEL

Equation [11] shows that admixture distribution in a flow is due to two factors, i.e. convective transfer and turbulent diffusion. Both processes may be taken into account in a one-dimensional diffusion model by including effective parameters the values of which integrally depend on a flow velocity and local transfer coefficients.

To build such a model we make use of the procedure described in Maron (1978).

Assume that a deposit layer thickness is $h \ll H$ and that a deposit layer does not considerably affect the velocity distribution. Formulate condition [13] at $z=0$:

$$\left(\epsilon\frac{\partial S}{\partial z}+aS\right)_{z=0}=(v_{2z}S)_{z=0}. \qquad [16]$$

Include the following dimensionless parameters and variables

$$\tau=\frac{\epsilon_0 t}{H^2},\ \xi=\frac{x}{H},\ \eta=\frac{z}{H},\ c=\frac{S}{S_0},\ \chi=\frac{\epsilon}{\epsilon_0},$$
$$Y=\frac{UH}{\epsilon_0},\ \mathrm{Pe}=\frac{aH}{\epsilon_0},\ \Phi=\frac{V_x}{U},\ \alpha=v_{2z}/a,\ (a\neq 0). \qquad [17]$$

ϵ_0 and S_0 are typical values of transfer coefficient and concentration; U is a flow velocity.

Equation [11] and boundary conditions [14] and [16] in terms of dimensionless variables will be expressed as follows:

$$\frac{\partial c}{\partial \tau}-\mathrm{Pe}\frac{\partial c}{\partial \eta}+Y\Phi(\eta)\frac{\partial c}{\partial \xi}=\frac{\partial}{\partial \eta}\left(\chi(\eta)\frac{\partial c}{\partial \eta}\right) \qquad [18]$$

$$\left(\chi(\eta)\frac{\partial c}{\partial \eta}+\mathrm{Pe}\,c\right)_{\eta=1}=0,\ \left[\chi(\eta)\frac{\partial c}{\partial \eta}+(1-\alpha)\mathrm{Pe}\,c\right]_{\eta=0}=0. \qquad [19]$$

Average particle cross-section concentration is

$$\theta(\tau, \xi) = \int_0^1 c(\tau, \xi, \eta)\,\mathrm{d}\eta. \tag{20}$$

Integrating the terms of [18] across channel cross-section and taking into account [19] we get:

$$\begin{gathered}\frac{\partial\theta}{\partial\tau} + Y\frac{\partial\theta}{\partial\xi} + Y\int_0^1 (\Phi(\eta) - 1)\frac{\partial c}{\partial\xi}\,\mathrm{d}\eta + \alpha\,\mathrm{Pe}\,c/_{\eta=0} = 0,\\ \theta = \theta(\tau, \xi),\ \tau > 0,\ -\infty < \xi < \infty.\end{gathered} \tag{21}$$

Let the function $\psi(\tau, \xi, \eta) = c(\tau, \xi, \eta) - \theta(\tau, \xi)$ be introduced.

Combining the terms of [18] and [21] we get an equation for $\psi = \psi(\tau, \xi, \eta)$:

$$\begin{aligned}\frac{\partial\psi}{\partial\tau} - \mathrm{Pe}\frac{\partial\psi}{\partial\eta} - \frac{\partial}{\partial\eta}\left(\chi(\eta)\frac{\partial\psi}{\partial\eta}\right) = &- Y(\Phi(\eta) - 1)\frac{\partial\theta}{\partial\xi} - Y(\Phi(\eta) - 1)\frac{\partial\psi}{\partial\xi}\\ &- Y\frac{\partial\psi}{\partial\xi} + Y\int_0^1 (\Phi(\eta) - 1)\frac{\partial\psi}{\partial\xi}\,\mathrm{d}\eta + \alpha\,\mathrm{Pe}\,(\psi/_{\eta=0} + \theta).\end{aligned} \tag{22}$$

To find a solution use the successive approximations method and substitute $\psi = 0, \eta \in (0, 1)$ into the r.h.s. of [22]. Such an approximation is based on the fact that for times which are much greater than a diffusion constant H^2/ϵ_0 admixture concentration in depth is almost equalised and only slight deviations of local concentration from the average value are observed. These deviations are due to heterogeneous (because of a velocity profile) admixture convictive transfer.

As a result of the substitution we have:

$$\frac{\partial\psi}{\partial\tau} - \mathrm{Pe}\frac{\partial\psi}{\partial\eta} - \frac{\partial}{\partial\eta}\left(\chi(\eta)\frac{\partial\psi}{\partial\eta}\right) = - Y(\Phi(\eta) - 1)\frac{\partial\theta}{\partial\xi} + \alpha\,\mathrm{Pe}\,(\psi/_{\eta=0} + \theta). \tag{23}$$

The boundary conditions for function ψ are:

$$\begin{gathered}\left(\chi(\eta)\frac{\partial\psi}{\partial\eta} + \mathrm{Pe}\,\psi\right)_{\eta=1} = -\mathrm{Pe}\,\theta,\\ \left[\chi(\eta)\frac{\partial\psi}{\partial\eta} + (1-\alpha)\,\mathrm{Pe}\,\psi\right]_{\eta=0} = -(1-\alpha)\mathrm{Pe}\,\theta.\end{gathered} \tag{24}$$

Suppose the initial condition be $\psi(0, \xi, \eta) = 0$. Introduce the following functions:

$$r(\eta) = \exp \mathrm{Pe}\int_0^\eta \frac{\mathrm{d}\eta'}{\chi(\eta')},\quad P(\eta) = \chi(\eta)\cdot r(\eta). \tag{25}$$

For these functions an η = differentiation operator in [23] may be written in a self-conjugate form:

$$L(\psi) = \frac{1}{r(\eta)}\frac{\partial}{\partial\eta}\left[P(\eta)\frac{\partial\psi}{\partial\eta}\right]. \tag{26}$$

Using this operator we represent [23] as:

$$L(\psi) = \frac{\partial \psi}{\partial \tau} + F,$$

$$F = Y(\Phi(\eta) - 1)\frac{\partial \theta}{\partial \xi} - f(\tau, \xi), \quad [27]$$

$$f = \alpha \, \mathrm{Pe}(\psi/_{\eta=0} + \theta).$$

The solution of [27] and subsequent approximations to solution of [22] are sought as a series with respect to eigenfunctions $X_n(\eta)$:

$$\psi(\tau, \xi, \eta) = \sum_{n=1}^{\infty} \frac{u_n(\tau, \xi)}{\|X_n\|^2} X_n(\eta), \; u_n = \int_0^1 r\psi X_n \, \mathrm{d}\eta, \; \|X_n\|^2 = \int_0^1 r X_n^2 \, \mathrm{d}\eta. \quad [28]$$

The eigenfunctions $X_n(\eta)$ satisfy the following Sturm–Liouville problem:

$$[P(\eta)X_n'(\eta)]' + \lambda_n^2 r(\eta)X_n(\eta) = 0, \; X_n = X_n(\eta), \; 0 < \eta < 1,$$
$$[\chi(\eta)X_n'(\eta) + \mathrm{Pe}\, X_n(\eta)]_{\eta=0;1} = 0. \quad [29]$$

Here λ_n is an eigenvalue of the mentioned problem.

We substitute a series [28] into an integral form of [21]:

$$\frac{\partial \theta}{\partial \tau} + Y\frac{\partial \theta}{\partial \xi} + Y \sum_{n=1}^{\infty} \frac{a_n}{\|X_n\|^2} \frac{\partial u_n}{\partial \xi} + f(\tau, \xi) = 0$$

$$\theta = \theta(\tau, \xi), a_n = \int_0^1 (\Phi(\eta) - 1)X_n(\eta)\, \mathrm{d}\eta. \quad [30]$$

To determine unknown functions $u_n(\tau, \xi)$ we should multiply the left-hand and right-hand parts of [27] by $rX_n(\eta)$ and integrate with respect to η from 0 to 1. Taking into consideration boundary conditions [24] and [29] we get an equation for a function $u_n(\tau, \xi)$:

$$\frac{\partial u_n}{\partial \tau} + \lambda_n^2 u_n = \mathrm{Pe}\, b_n \theta - Y c_n \frac{\partial \theta}{\partial \xi} - d_n f_n(\tau, \xi), \; u_n = u_n(\tau, \xi),$$

$$\tau > 0, \; -\infty < d < \xi < b < \infty, \; b_n = X_n(0) - X_n(1)r(1), \quad [31]$$

$$c_n = \int_0^1 r(\Phi(\eta) - 1)X_n \, \mathrm{d}\eta, \; d_\eta = X_n(0) + \frac{1}{\lambda_n^2}\mathrm{Pe}\, b_n.$$

The solution of [31] for initial condition $u_n(0, \xi) = 0$ is

$$u_n(\tau, \xi) = b_n \mathrm{Pe} \int_0^\tau \exp[-\lambda_n^2(\tau - s)]\theta(s, \xi)\, \mathrm{d}s - c_n Y \int_0^\tau \exp[-\lambda_n^2(\tau - s)]$$

$$\times \frac{\partial \theta(s, \xi)}{\partial \xi} \mathrm{d}s - d_n \int_0^\tau \exp[-\lambda_n^2(\tau - s)] f(s, \xi)\, \mathrm{d}s \quad [32]$$

substitute [32] into [30]

$$\frac{\partial \theta}{\partial \tau} + Y\frac{\partial \theta}{\partial \xi} + Y\,\mathrm{Pe}\sum_{n=1}^{\infty}\frac{a_n b_n}{\|X_n\|^2}\int_0^{\tau}\exp\left[-\lambda_n^2(\tau - s)\right]\frac{\partial \theta(s,\xi)}{\partial \xi}\,\mathrm{d}s$$
$$= Y^2\sum_{n=1}^{\infty}\frac{a_n c_n}{\|X_n\|^2}\int_0^{\tau}\exp\left[-\lambda_n^2(\tau - s)\right]\frac{\partial^2 \theta(s,\xi)}{\partial \xi^2}\,\mathrm{d}s \qquad [33]$$
$$+ Y\sum_{n=1}^{\infty}\frac{a_n d_n}{\|X_n\|^2}\int_0^{\tau}\exp\left[-\lambda_n^2(\tau - s)\right]\frac{\partial f(s,\xi)}{\partial \xi}\,\mathrm{d}s.$$

Equation [33] includes two unknown functions, i.e. $\theta(\tau, \xi)$ and $f(\tau, \xi)$. These two functions are interconnected:

$$f(\tau,\xi) = \alpha\,\mathrm{Pe}\,(\psi/_{\eta=0}T + \theta) = \alpha\,\mathrm{Pe}\left[\theta + \sum_{n=1}^{\infty}\frac{u_n(\tau,\xi)}{\|X_n\|^2}X_n(0)\right]$$
$$= \alpha\,\mathrm{Pe}\left\{\theta + \mathrm{Pe}\sum_{n=1}^{\infty}\frac{b_n X_n(0)}{\|X_n\|^2}\int_0^{\tau}\exp\left[-\lambda_n^2(\tau - s)\right]\theta(s,\xi)\,\mathrm{d}s\right. \qquad [34]$$
$$- Y\sum_{n=1}^{\infty}\frac{c_n X_n(0)}{\|X_n\|^2}\int_0^{\tau}\exp\left[-\lambda_n^2(\tau - s)\right]\frac{\partial \theta(s,\xi)}{\partial \xi}\,\mathrm{d}s$$
$$\left. - \sum_{n=1}^{\infty}\frac{d_n X_n(0)}{\|X_n\|^2}\int_0^{\tau}\exp\left[-\lambda_n^2(\tau - s)\right].\right.$$

Two integro-differential equations [33] and [34] constitute a one-dimensional model to determine a mean flow cross-section concentration of admixture. The model takes into account delay effects in equalizing concentration across the channel cross-section, particle sedimentation and formation of deposit layer. Based on this model one may solve problems for different initial and boundary conditions which once solved help to determine mean concentration $\theta(\tau, \xi)$. A cross-section concentration distribution may be determined from the following relationship:

$$c(\tau,\xi,\eta) = \theta(\tau,\xi) + \sum_{n=1}^{\infty}\frac{u_n(\tau,\xi)}{\|X_n\|^2}X_n(\eta). \qquad [35]$$

The coefficients in [33]–[35] are obtained based on a solution of Sturm–Liouville problem [29].

6. STURM–LIOUVILLE PROBLEM [29]

If we assume that diffusion coefficient is constant across a tube cross-section, i.e. $\chi(\eta) = 1$, that corresponds to laminar flow conditions, a solution of Sturm–Liouville problem is following:

$$X_n(\eta) = \exp\left(-\frac{1}{2}\mathrm{Pe}\,\eta\right)\left(\cos n\pi\eta - \frac{\mathrm{Pe}}{2n\pi}\sin n\pi\eta\right),$$
$$\|X_n\|^2 = 1 + \frac{\mathrm{Pe}^2}{4n^2\pi^2},\quad \lambda_n^2 = \frac{1}{4}\mathrm{Pe}^2 + n^2\pi^2. \qquad [36]$$

In case of a turbulent flow a diffusion coefficient $\chi(\eta)$ changes across the channel cross-section, therefore it is impossible to find an analytical solution of Sturm–Liouville problem [29].

For a numerical solution of this problem the Gauchy conditions are given at $\eta = 0$:

$$X_n(0) = 1, \; X_n'(0) = -\frac{\mathrm{Pe}}{X_n(0)}. \tag{37}$$

It may be shown that $X_n(0) \neq 0$. Further on there was used an iterative process of choosing such values of λ_n^2 at which the solution of [29] satisfies a condition at $\eta = 1$.

7. THE COEFFICIENTS [31]

In a laminar flow the velocity profile is

$$\Phi(\eta) = 6\eta(1-\eta) \tag{38}$$

Coefficients a_n, b_n, c_n and d_n are:

$$\begin{aligned}
a_n &= -\frac{6}{\lambda_n^2}\left\{1 + (-1)^n \exp\frac{1}{2}\mathrm{Pe} - \frac{2\,\mathrm{Pe}}{\lambda_n^2}\left[1 - (-1)^n \exp\frac{1}{2}\mathrm{Pe}\right]\right\},\\
b_n &= 1 - (-1)^n \exp\frac{1}{2}\mathrm{Pe},\\
c_n &= \frac{1}{\lambda_n^2}\left[\mathrm{Pe} - 6 - (\mathrm{Pe}+6)(-1)^n \exp\frac{1}{2}\mathrm{Pe}\right]\\
&\quad + \frac{6\,\mathrm{Pe}}{\lambda_n^4}\left[\mathrm{Pe} - 4 - (\mathrm{Pe}+4)(-1)^n \exp\frac{1}{2}\mathrm{Pe}\right] + \frac{12\,\mathrm{Pe}^3}{\lambda_n^6}\left[1 - (-1)^n \exp\frac{1}{2}\mathrm{Pe}\right],\\
d_n &= 1 + \frac{\mathrm{Pe}}{\lambda_n^2}\left[1 - (-1)^n \exp\frac{1}{2}\mathrm{Pe}\right].
\end{aligned} \tag{39}$$

When determining these coefficients for a turbulent flow those approximations of velocity profile and turbulent diffusion coefficient were used that were proposed by Reichardt (1951):

$$\begin{gathered}
u(\eta) = \frac{u_*}{\kappa}\ln\left[(1+\kappa\delta) + \frac{1.5(1+\omega)}{1+2\omega^2}\right] + 7.8\left[1 - \exp\left(-\frac{\delta}{\delta_1}\right) + \frac{\delta}{\delta_1}\exp(-0.33\delta)\right]\\
\delta_1 = 11.1, \; \kappa = 0.4, \; \omega = |1-2\eta|, \; \delta = (1-\omega)\,\mathrm{Re}_*,\\
\mathrm{Re}_* = \frac{u_* H}{2\nu}, \; \epsilon = D + \frac{\kappa H u_*}{12}(1-\omega^2)(1+2\omega^2)
\end{gathered} \tag{40}$$

where D is a molecular diffusion coefficient.

Using [40] we may determine U and ϵ_0.

8. ASYMPTOTIC FOR $t \to \infty$

Dispersion equation

$$\int_0^\tau \exp[-\lambda_n^2(\tau - s)]\theta(s,\xi)\,\mathrm{d}s \Rightarrow \frac{\theta(\tau,\xi)}{\lambda_n^2}. \tag{41}$$

Taking into account [41] we may easily determine $f(\tau, \xi)$ using [34] and then substitute it into [33]. As a result we get an equation which contains only one function $\theta(\tau, \xi)$:

$$\frac{\partial \theta}{\partial \tau} + V\frac{\partial \theta}{\partial \xi} + \Gamma\theta = K\frac{\partial^2 \theta}{\partial \xi^2}, \quad \theta = \theta(\tau, \xi), \ \tau > 0, \ -\infty < d < \xi < b < \infty. \qquad [42]$$

In this equation

$$V = Y\left(1 + \mathrm{Pe}\sum_{n=1}^{\infty}\frac{a_n b_n}{\lambda_n^2\|X_n\|^2} - \frac{\alpha g_2 d_n}{g} - \frac{\alpha g_1}{g}\right),$$

$$K = Y^2\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^2\|X_n\|^2}\left(c_n - \frac{\alpha\,\mathrm{Pe}\,g_1 d_n}{g}\right),$$

$$\Gamma = \alpha\,\mathrm{Pe}\frac{g_2}{g},$$

$$g = 1 + \alpha\,\mathrm{Pe}\left[\sum_{n=1}^{\infty}\frac{X_n^2(0)}{\lambda_n^2\|X_n\|^2} + \mathrm{Pe}\sum_{n=1}^{\infty}\frac{b_n X_n(0)}{\lambda_n^4\|X_n\|^2}\right],$$

$$g_1 = \sum_{n=1}^{\infty}\frac{c_n X_n(0)}{\lambda_n^2\|X_n\|^2},$$

$$g_2 = \mathrm{Pe}\sum_{n=1}^{\infty}\frac{b_n X_n(0)}{\lambda_n^2\|X_n\|^2}. \qquad [43]$$

The parameters of [42] may be also found using the method described in Taylor's papers. We get the following expressions:

$$V = 1 + \left(\int_0^1\frac{\Phi(\eta) - 1}{r(\eta)}\,\mathrm{d}\eta\right)\left(\int_0^1\frac{\mathrm{d}\eta}{r(\eta)}\right)^{-1} - \alpha\,\mathrm{Pe}\,(\psi_1 + \psi_2\psi_3),$$

$$K = \frac{1}{\mathrm{Pe}}\left\{\int_0^1\Phi(\eta)\frac{\beta(\eta)}{r(\eta)}\,\mathrm{d}\eta - V\int_0^1\frac{\beta(\eta)}{r(\eta)}\,\mathrm{d}\eta\right\} - \alpha\,\mathrm{Pe}\,\psi_1\psi_3,$$

$$\Gamma = \alpha\,\mathrm{Pe}\,\psi_2, \quad \beta = \int_0^\eta[\Phi(\eta) - 1]r(\eta)\,\mathrm{d}\eta,$$

$$\psi_1 = -\frac{1}{\psi_0}\frac{\int_0^1\frac{\beta(\eta)}{r(\eta)}\,\mathrm{d}\eta}{\mathrm{Pe}\int_0^1\frac{\mathrm{d}\eta}{r(\eta)}}, \quad \psi_2 = \frac{1}{\psi_0}\left(\int_0^1\frac{\mathrm{d}\eta}{r(\eta)}\right)^{-1},$$

$$\psi_3 = \int_0^1[\Phi(\eta) - 1]\kappa_1(\eta)\,\mathrm{d}\eta, \quad \kappa_1 = \eta + \frac{1}{r(\eta)} - \frac{R(\eta)}{r(\eta)},$$

$$\psi_0 = 1 + \alpha\left[\frac{1}{2} - \int_0^1\frac{\mathrm{d}\eta}{r(\eta)} + \int_0^1\frac{R(\eta)}{r(\eta)}\,\mathrm{d}\eta\right]\left(\int_0^1\frac{\mathrm{d}\eta}{r(\eta)}\right)^{-1},$$

$$R(\eta) = \int_0^\eta r(\eta)\,\mathrm{d}\eta. \qquad [44]$$

These formulas do not require a solution of Sturm–Liouville problem.

Equation [42] contains three dimensionless effective parameters, i.e. K = a longitudinal diffusion coefficient accounting for the processes of local diffusion, convective transfer and particle deposition, V = convective transfer velocity and Γ = a parameter accounting for particle separation from the flow as a result of their deposition on the channel bottom.

If particles do not deposit ($a = 0$) and stick to the channel bottom ($|\alpha = 0|$), [36] transfers into Taylor equation. If particles deposit ($a \neq 0$) but do not remain still on the channel bottom ($|\alpha = 0|$) [36] transfers into a convective diffusion equation ($|\Gamma = 0|$) derived in Lurie & Maron (1979).

Equation [42] gives an asymptotic distribution of admixture particles concentration when $\tau \to \infty$. In a turbulent flow due to intensive mixing such a distribution is observed very soon after the process has initiated ($t \sim H^2/\epsilon_0$). Therefore [42] solution in fact may be used from the beginning of the process. Equation [42] contains a term $\Gamma\theta$ and is analogous to a longitudinal diffusion equation for radioactive admixture (Maron 1978). But in the case considered this term characterised particle disappearance from a flow as a result of deposition on the channel bottom.

The dispersion of particles injected into a flow at the entrance of channel may be represented as:

$$\sigma^2 = 2K\tau \tag{45}$$

where σ denotes a dispersion.

The dispersion-time law does not differ from the linear law.

9. EQUATION [42] PARAMETERS DETERMINATION

Plots of V, K, Γ as functions of Pe number are given in figures 1–6. Based on these plots we may conclude about the effect of α- and Pe-parameters on values of the coefficients in [42].

It follows from the plots that a convective transfer velocity V, when $\alpha \neq 0$ and Pe $\neq 0$ differ slightly from a mean flow velocity. The effective diffusion coefficient K depends more on α- and Pe-parameters.

10. ADMIXTURE CONCENTRATION PROFILES FOR A TURBULENT FLOW

Let distribution of mean admixture concentration described by the solution of [42] satisfy the following conditions:

$$\theta(0, \xi) = 0, \ \theta(\tau, 0) = 1, \ \theta(\tau, \infty) = 0. \tag{46}$$

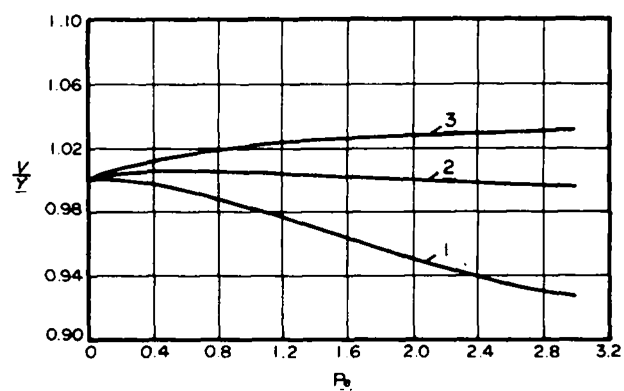

Figure 1. Laminar flow convective transfer velocity vs particle settling velocity at different values of α-parameter. Re = 10^2, Sc = 10^2. (1) $\alpha = 0$; (2) $\alpha = 0.5$; (3) $\alpha = 1$.

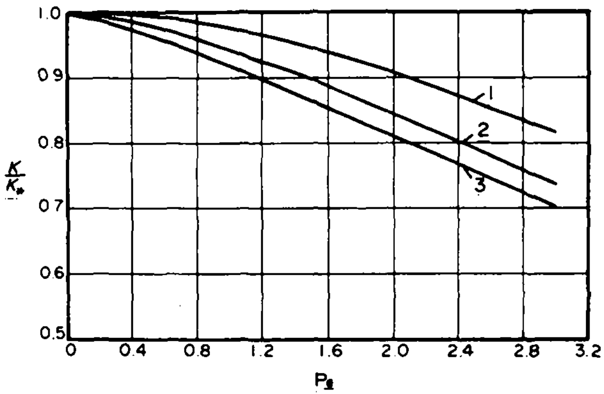

Figure 2. Laminar flow longitudinal diffusion coefficient vs particle settling velocity at different values of α. $Re = 10^2$, $Sc = 10^2$, (1) $\alpha = 0$; (2) $\alpha = 0.5$; (3) $\alpha = 1$. $K_* =$ Taylor coefficient for $Re = 10^2$, $Sc = 10^2$.

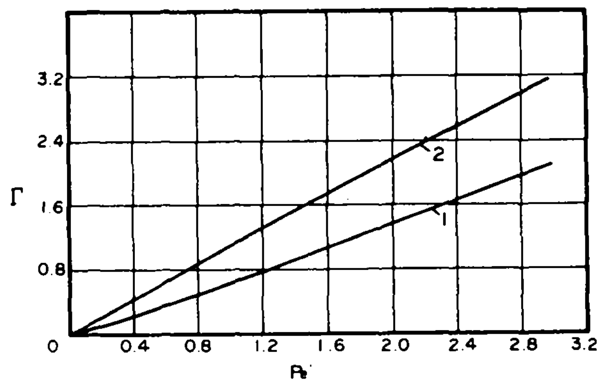

Figure 3. Γ-value vs Pe-value (laminar flow). $Re = 10^2$, $Sc = 10^2$, (1) $\alpha = 0.5$; (2) $\alpha = 1$.

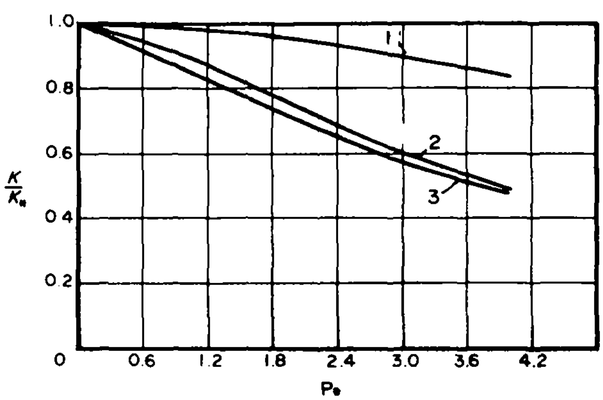

Figure 4. Turbulent flow longitudinal diffusion coefficient vs particle selling velocity. $Re = 10^4$, $Sc = 10^2$. (1) $\alpha = 0$; (2) $\alpha = 0.5$; (3) $\alpha = 1$. K_* – Taylor coefficients for $Re = 10^4$, $Sc = 10^2$.

The solution of [42] which corresponds to the above conditions may be represented as follows:

$$\theta(\tau, \xi) = \frac{2}{\sqrt{\pi}} \exp\left(\frac{V\xi}{2K}\right) \int_{\xi/2\sqrt{K\tau}}^{\infty} \exp\left[-\left(\Gamma + \frac{V^2}{4K}\right)\frac{\xi^2}{4Kx} - x^2\right] dx. \qquad [47]$$

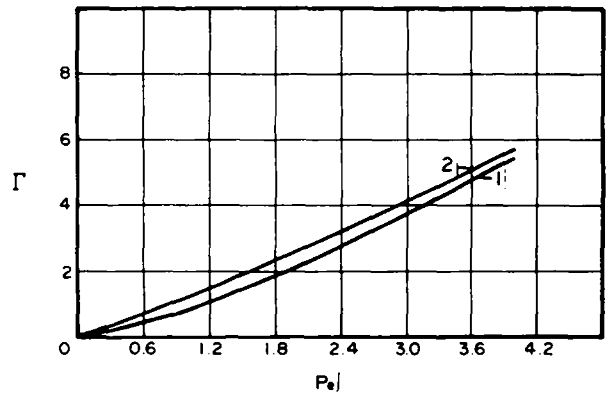

Figure 5. Γ-value vs Pe-value (turbulent flow). Re = 10^4, Sc = 10^2. (1) $\alpha = 0.5$; (2) $\alpha = 1$.

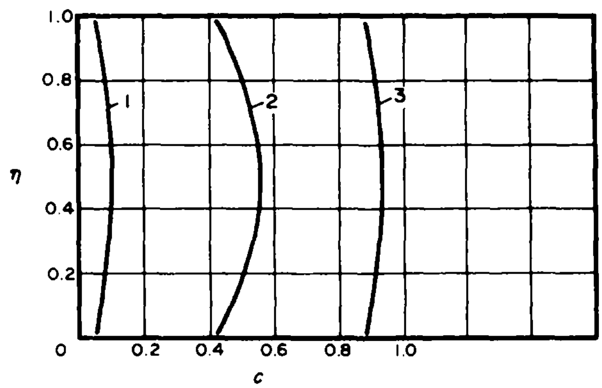

Figure 6. Concentration profile for particle admixture in a turbulent flow. Re = 10^4, Sc = 10^2 at Pe = 0 and $\alpha = 0$ in cross-section $\zeta = V$. (1) $\tau = 0.95$; (2) $\tau = 1$; (3) $\tau = 1.05$.

Based on this solution we may determine a local concentration using [35] in which

$$u_n(\tau, \xi) = -Y\frac{1}{\lambda_n^2}\left(c_n - \frac{\alpha\,\mathrm{Pe}\,g_1}{g}d_n\right)\frac{\partial\theta}{\partial\xi} + \frac{\mathrm{Pe}}{\lambda_n^2}\left(b_n - \frac{\alpha g_2}{g}d_n\right)\theta. \qquad [48]$$

The eigenfunctions $X_n(\eta)$ may be determined from a solution of Sturm–Liouville problem for a certain set of parameters which characterise a turbulent flow.

Figures 7–9 gives admixture concentration profiles at different values of α and Pe for Reynolds number $\mathrm{Re} = 10^4$ and Schmidt number $\mathrm{Sc} = 10^2$. The cross-section for which these profiles are determined at different time instants, is at a distance of $\xi = V$ from the channel entrance. The plots show that admixture deposition at $\alpha = 0$ result in a marked assymmetry of a profile form, since admixture particles accumulate the channel bottom. If $\alpha \neq 0$, a concentration profile is more symmetric as a portion of admixture particles sticks to the channel bottom and a deposit layer increases, its thickness satisfies the following equation:

$$\frac{\partial\Delta}{\partial\tau} = \frac{\alpha\,\mathrm{Pe}c/_{\eta=0}}{c_* - c/_{\eta=0}} \approx \frac{\alpha\,\mathrm{Pe}}{c_*}\left(\frac{g_2}{g}\theta - Y\frac{g_1}{g}\frac{\partial\theta}{\partial\xi}\right) \qquad [49]$$

where $\Delta = h/H$. When derivating this relationship it was assumed that $c_* \gg c/_{\eta=0}$.

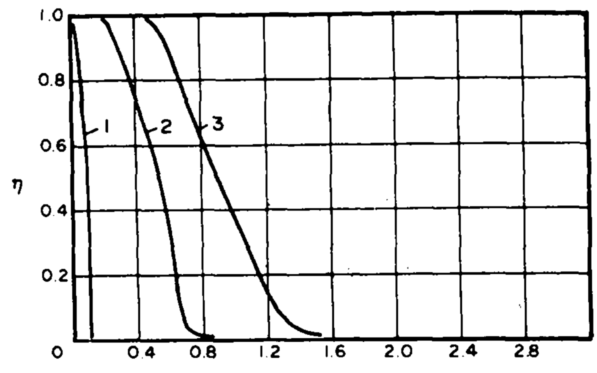

Figure 7. Concentration profile for particle admixture in a turbulent flow. $Re = 10^4$, $Sc = 10^2$ at $Pe = 1$ and $\alpha = 0$ in cross-section $\xi = V$. (1) $\tau = 0.95$; (2) $\tau = 1$; (3) $\tau = 1.05$.

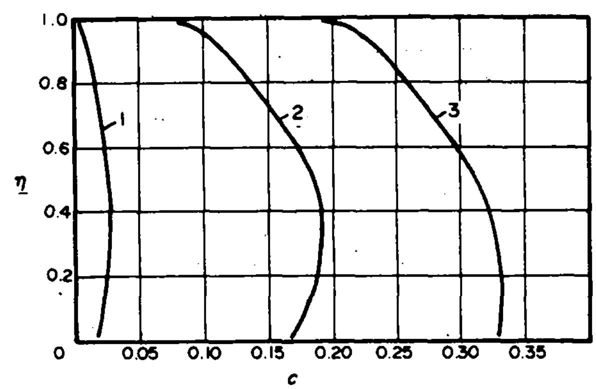

Figure 8. Concentration profile for particle admixture in a turbulent flow. $Re = 10^4$, $Sc = 10^2$ at $Pe = 1$ and $\alpha = 1$ in cross-section $\zeta = V$. (1) $\tau = 0.95$; (2) $\tau = 1$; (3) $\tau = 1.05$.

Another formula may be used to determine $c(\tau, \xi, \eta)$. This formula is derived using Taylor method. We get:

$$c(\tau,\xi,\eta) = \theta(\tau,\xi)\left[\alpha\psi_2(1-\kappa_1(\eta)) + \frac{\psi_2}{r(\eta)}\right] - Y\frac{\partial\theta}{\partial\xi}\left\{\alpha\psi_1(1-\kappa_1(\eta)) + \frac{\psi_1}{r(\eta)} - \frac{1}{Pe}\left[\psi(\eta) - \frac{\beta(\eta)}{r(\eta)}\right]\right\}, \quad [50]$$

where

$$\psi(\eta) = \int_0^\eta [\Phi(\eta) - 1]\,d\eta.$$

11. CONCLUSIONS

The equations proposed may be used to describe admixture propagation which occurs in rivers and channels.

To research aerosol dispersion it is necessary to develop a theory in which the effect of particle adhesion to tube inner surface is considered. We may assume that $\alpha \neq 0$ and

Pe = 0 while describing aerosol transfer when particle motion time in a tube is small compared to a characteristic deposition time.

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